1.7 Appendix Chapter 1 Infinite Products

Unproved Result 1 *If* $\prod_{n=1}^{\infty} u_n$ *converges, then the product of inverses,* $\prod_{n=1}^{\infty} u_n^{-1}$*, converges.*

Proof Let $p_n = \prod_{i=1}^{n} u_i$ so that $p_n \to p$ with $p_n \neq 0$ for all $n \geq 1$ and $p \neq 0$. But then by the Quotient Law for Sequences $p_n^{-1} \to p^{-1}$ with $p^{-1} \neq 0$. This is the definition of $\prod_{n=1}^{\infty} u_n^{-1}$ converging. ■

Unproved Result 2 *If the* ***series*** $\sum_{n=1}^{\infty} |a_n|$ *is convergent (where the* a_n *are real or complex and* $a_n \neq -1$ *for all* n*, then the* ***infinite product*** $\prod_{n=1}^{\infty} (1 + a_n)$ *converges in that the limit*

$$\lim_{N \to \infty} \prod_{n=1}^{N} (1 + a_n)$$

exists and is ***non-zero****.*

Proof Let $p_n = \prod_{i=1}^{n} (1 + a_i)$ for $n \geq 1$ and $S = \sum_{n=1}^{\infty} |a_n|$ which, by assumption, converges.

For the first step use $1 + x \leq \exp(x)$ for $x > 0$. Then

$$|p_n| \leq \prod_{i=1}^{n} (1 + |a_i|) \leq \prod_{i=1}^{n} \exp(|a_i|) = \exp\left(\sum_{i=1}^{n} |a_i| \right) \leq \exp(S), \qquad (16)$$

for all $n \geq 1$. Next observe that $p_n = (1 + a_n) p_{n-1}$ for $n \geq 2$. Consider

$$p_n - p_1 = \sum_{i=2}^{n} (p_i - p_{i-1}) = \sum_{i=2}^{n} a_i p_{i-1}. \qquad (17)$$

Then

$$\begin{aligned} \sum_{i=2}^{n} |a_i p_{i-1}| &\leq \sum_{i=2}^{n} |a_i| e^S \quad \text{by } (16) \\ &\leq Se^S \end{aligned}$$

Thus the series $\sum_{i=2}^{n} a_i p_{i-1}$ converges (absolutely) and so, by (17), $\lim_{n \to \infty} p_n$ exists.

To show that p_n converges to a *non-zero* limit p the **trick** is to show that $\prod_{n=1}^{\infty} (1 + a_n)^{-1}$ converges to q where $pq = 1$. Then $p \neq 0$.

The idea is to use the same result as above that gave the convergence of $\prod_{n=1}^{\infty}(1+a_n)$. So rewrite $(1+a_n)^{-1} = 1+b_n$ and try to show that $\sum_{n=1}^{\infty}|b_n|$ converges. But

$$b_n = \frac{1}{1+a_n} - 1 = -\frac{a_n}{1+a_n}.$$

The assumption $\sum_{n=1}^{\infty}|a_n|$ converges implies $a_n \to 0$ so there exists $N \geq 1$ such that $|a_n| < 1/2$, i.e. $|1+a_n| \geq 1/2$ and thus $|b_n| \leq 2|a_n|$ for all $n \geq N$. Then $\sum_{n=1}^{\infty}|b_n|$ converges by comparison with $\sum_{n=1}^{\infty}2|a_n|$. The result follows. ■